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Bifurcation and Stability of Stationary Solutions of the Fitz-Hugh–Nagumo Equations

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1. INTRODUCTION

The Fitz-Hugh–Nagumo equations have been of some interest to both mathematicians and theoretical biologists for several years. The reason for this stems from the fact that they can be considered as a simpler model for the celebrated Hodgkin–Huxley equations, in that they exhibit many of the features of this latter system. Indeed, mathematicians have studied them because their structure is different from the usually encountered equations in physics, and they therefore admit solutions with less familiar properties: homoclinic travelling waves, threshold effects, etc.

The equations can be written as

$$\begin{aligned} v_t &= v_{xx} + f(v) - u \\ u_t &= \delta v - \gamma u \qquad (x, t) \in \Omega \times \mathbb{R}_+ \subset \mathbb{R} \times \mathbb{R}_+ \end{aligned} \tag{1.1}$$

where δ and γ are positive constants, and $f(v)$ has the qualitative shape of a cubic polynomial having two positive roots, and satisfies $f(0)=0$, $f'(0)<0$. Furthermore, the area of the "hill" exceeds that of the "valley." For simplicity, one usually takes

$$f(v) = -(v-c)(v-1)\,V, \qquad 0 < c < \tfrac{1}{2}. \tag{1.2}$$

In this paper, we shall concern ourselves with bounded spatial regions $\Omega = \{|x| < L\}$; this requires that we take v to satisfy boundary conditions at $\pm L$, and we shall assume that v satisfies either homogeneous Dirichlet or Neumann boundary conditions.

The system (1.1) admits a special class of solutions called stationary solutions. These are solutions which are independent of t, and thus they satisfy the equations $v_{xx} + f(v) - u = 0$, $\delta v - \gamma u = 0$. Our main point in this

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paper is to discuss the stability and bifurcation of these solutions. In this context, we shall take as bifurcation parameters any of the quantities δ, γ, or L, and we shall see here that interesting cases arise when δ/γ is small, and L is large.

Our point of view will be to consider the system (1.1) as an ordinary differential equation, $U' = A(U)$, in an infinite dimensional function space; see [3, 4, 5]. The equations thus define a semi-flow on this space, and the "rest points" of A correspond to the stationary solutions of (1.1). We shall find all of the rest points, and show that they are "non-degenerate." Using certain topological techniques (see [2, 4, 5, 10]), we will be able to compute the (generalized) Morse indices (i.e., the dimensions of the unstable manifolds) of each rest point. For a range of parameter values, we will show that the system (1.1) is "gradient-like," and this fact will enable us to describe the complete solution space. As the parameters change, we will study the bifurcation of stationary solutions; in particular, we will prove the existence of time-periodic solutions.

As was shown in [8], there is a fixed rectangle $R = R(\delta, \gamma)$ in $v-u$ space, containing $(0, 0)$, having the property that all solutions of (1.1) tend to R in $L^\infty(|x| < L)$, as $t \to \infty$; in other words, R is a global attracting region. It follows that R contains all stationary solutions, and thus the "potentials" which appear in the linearized equations (about these stationary solutions) are all bounded functions. It is this fact which will enable us to get some control on the spectra of the linearized operators and to thereby make applicable the aforementioned topological techniques.

We assume that the reader has some familiarity with the generalized Morse theory as developed in [2]; see also [10]. In particular, we use the notations $h(I)$ to denote the index of the isolated invariant set I, and Σ^k to denote the pointed k-sphere.

2. Background and Formulation of the Problem

A. *The Equations*

We consider Eqs. (1.1) on the domain $|x| < L$, together with one of the following boundary conditions:

$$v(\pm L, t) = 0, \qquad t > 0, \tag{2.1$_\text{D}$}$$

$$v_x(\pm L, t) = 0, \qquad t > 0. \tag{2.1$_\text{N}$}$$

These will be referred to as (homogeneous) Dirichlet, and Neumann conditions, respectively. In addition, we assume that v and u are prescribed initially, i.e.,

$$(v(x, 0), u(x, 0)) = (\bar{v}(x), \bar{u}(x)), \qquad |x| \leqslant L, \tag{2.2}$$

where $\bar{v}$ and $\bar{u}$ are bounded smooth functions. As is shown in [1 or 10], the problem (1.1), (2.1), (2.2) has a unique bounded solution defined for all $t>0$.

We turn now to the stationary equations; they are

$$v''+f(v)-u=0, \qquad \delta v-\gamma u=0, \qquad |x|<L, \tag{2.3}$$

together with one of the corresponding boundary conditions

$$v(\pm L)=0. \tag{2.4$_{\rm D}$}$$

$$v'(\pm L)=0. \tag{2.4$_{\rm N}$}$$

Notice that (2.3) can be written in the form

$$v''+f(v)-\delta v/\gamma=0, \qquad u=\delta v/\gamma; \tag{2.3$'$}$$

this reduces the problem of finding all of the steady state solutions to that of finding all solutions of a single second-order ordinary differential equation satisfying the boundary conditions (2.4).

As we have noted in the Introduction, there is a rectangle R in $v-u$ space, containing $(0,0)$, to which all solutions of (1.1), (2.1), (2.2), tend to, uniformly in x, as $t\to+\infty$. Thus all of the steady state solutions, must lie in R. This gives the following result.

PROPOSITION 2.1. *All solutions of* (1.1), (2.1) *tend, as* $t\to+\infty$ *to a bounded rectangle* R *in* $(v-u)$*-space. Thus there is an* $M>0$ *such that all solution of* (2.3), (2.4) *satisfy* $\|v\|_\infty+\|u\|_\infty\leqslant M$.

B. *Steady State Solutions of the Dirichlet Problem*

We consider first the Dirichlet problem (2.3), $(2.4)_{\rm D}$. It is easy to see that if δ/γ is sufficiently large, then $v\equiv 0\equiv u$ is the only solution. We shall thus take δ/γ so small that: (i) the function $f(v)-\delta v/\gamma$ has three (distinct) real roots $0<a<b$; and (ii)

$$\int_0^b (f(v)-\delta v/\gamma)\,dv>0. \tag{2.5}$$

In this case, the phase plane for the equation

$$v''+f(v)-\delta v/\gamma=0 \tag{2.6}$$

takes the form as depicted in Fig. 1. Obviously $v=v_0\equiv 0$ is always a solution of the Dirichlet problem.

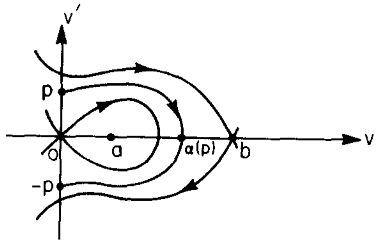

FIGURE 1

From the results of [12], we know that there is a number L^* such that (2.6), (2.4$_D$), has only $v \equiv 0$ as a solution if $L < L^*$; has exactly one non-constant solution if $L = L^*$; and has precisely two non-constant solutions if $L > L^*$. This information is contained in the global bifurcation diagram depicted in Fig. 2. If we take $L > L^*$, then we denote by v_1 and v_2 the two non-constant solutions, where $v_2'(-L) > v_1'(-L)$. It was shown in [3, 8], that the solutions $v_0 \equiv 0$ and v_2 are "attractors" for the associated parabolic equation

$$v_t = v_{xx} + f(v) - \delta v/\gamma, \qquad |x| < L,\ t > 0 \tag{2.7}$$

with boundary data (2.1). By this we mean that if the initial values $v(x, 0)$ are sufficiently close (in C^α) to either v_0 or v_2, then the corresponding solution of (2.7), (2.1) with this data, converges (in C^α) to the corresponding steady state solution. Similarly, v_1 is unstable, and has a 1-dimensional unstable manifold. It is also proved in [11] that all of these solutions are "non-degenerate" in the sense that zero is not in the spectrum of the linearized equations.

Translating these facts into analytical terms means that if we consider the linearized equations

$$\phi'' + q_i(x)\,\phi = \lambda\phi, \qquad \phi(\pm L) = 0, \tag{2.8}$$

where $q_i(x) = f'(v_i(x)) - \delta/\gamma$, $i = 0, 1, 2$, then the following theorem holds.

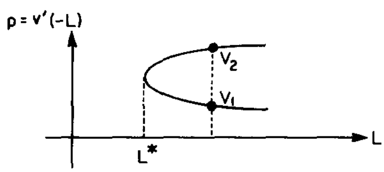

FIGURE 2

THEOREM 2.2. (a) *If $i=0$ or $i=2$, the problem* (2.8) *admits no non-zero solution if $\lambda \geqslant 0$.*

(b) *If $i=1$, the eigenvalue problem* (2.8) *has precisely one positive eigenvalue. All other non-zero solutions ϕ correspond to $\lambda<0$.*

C. *Steady State Solutions of the Neumann Problem*

We shall briefly consider the problem (2.3), (2.4_{N}). Again we choose δ/γ so small that the function $f(v)-\delta v/\gamma$ has three (distinct) real roots $0<a<b$; in this case we do not need condition (2.5).

It is obvious that the three constant functions 0, a, and b are always solutions of (2.6), (2.4_{N}). In [12] it was shown that there is a critical value of L, call it L^* again, such that if $L<L^*$, these are the only solutions. Furthermore, there is a sequence of real numbers $L_1=L^*<L_2<\cdots<L_n<\cdots$, $L_n\to\infty$, such that at each L_k, two solutions v_k^1, v_k^2 bifurcate out of a. These solutions were shown to be unstable solutions of the problem (2.7) with boundary data 2.1_{N}). In fact, for $i=1, 2$, v_k^i is non-degenerate, and has a $(k-1)$-dimensional unstable manifold (see [4]). Furthermore, the solutions $v\equiv 0$ and $v\equiv b$ are the only stable solutions of (2.9). Finally, the solution $v=a$ has a 1-dimensional unstable manifold if $0<L<L_1$, and a k-dimensional unstable manifold if $L_{k-1}<L<L_k$, $k>1$. Of course, 0 is in the spectrum of the linearization about $u\equiv a$ whenever $L=L_k$, $k\geqslant 1$.

3. THE DIRICHLET PROBLEM—PROPERTIES OF SOLUTIONS

We consider the Dirichlet problem (1.1), (2.1_{D}). The steady state solutions satisfy

$$v''+f(v)-\theta v=0, \qquad |x|<L, v(\pm L)=0, \tag{3.1}$$

where $\theta=\delta/\gamma$. As before, we choose θ so small, say $0\leqslant\theta<\bar{\theta}$, so that the function

$$f_\theta(v)=f(v)-\theta v \tag{3.2}$$

has three real roots $0<a_\theta<b_\theta$, and also that

$$\int_0^{b_\theta} f_\theta(v)\,dv>0. \tag{3.3}$$

For each such θ, we have the "time maps" T_θ defined by (see [12])

$$T_\theta(p)=\int_0^{\alpha_\theta(p)}\frac{dv}{\sqrt{2F_\theta(\alpha_\theta(p))-2F_\theta(v)}}, \qquad F_\theta'=f_\theta.$$

These functions denote the "time" an orbit in the phase plane for (3.1) takes to go from a point p on the v' axis to the point $\alpha_\theta(p)$ on the v axis (see Fig. 1). For $0 \leqslant \theta < \bar{\theta}$, the functions T_θ have exactly one minimum, p_θ. We define L_θ by

$$T_\theta(p_\theta) = L_\theta.$$

Concerning this quantity, we have the following lemma.

LEMMA 3.1. $\partial L_\theta/\partial\theta > 0$.

Proof. If $F' = f$, we can write

$$L_\theta = T_\theta(p_\theta) = \int_0^{\alpha_\theta(p_\theta)} \frac{dv}{\sqrt{2F(\alpha_\theta) - \theta\alpha_\theta^2 - 2F(v) + \theta v^2}} \equiv \Phi(\alpha_\theta, p_\theta, \theta),$$

so that

$$\frac{\partial L_\theta}{\partial\theta} = \frac{\partial\Phi}{\partial\alpha_\theta}\frac{\partial\alpha_\theta}{\partial\theta} + \frac{\partial\Phi}{\partial p_\theta}\frac{\partial p_\theta}{\partial\theta} + \frac{\partial\Phi}{\partial\theta}. \tag{3.4}$$

Now by definition of p_θ, $\partial\Phi/\partial p_\theta = (\partial T_\theta/\partial p)(p_\theta) = 0$, and if we set

$$S_\theta(\alpha_\theta) = \int_0^{\alpha_\theta} \frac{dv}{\sqrt{2F(\alpha_\theta) - \theta\alpha_\theta^2 - 2F(v) + \theta v^2}},$$

then $\partial T_\theta/\partial p = S'_\theta(\alpha_\theta)\, d\alpha_\theta/dp$, so that

$$0 = \frac{\partial T_\theta(p_\theta)}{\partial p} = S'_\theta(\alpha_\theta)\, d\alpha_\theta/dp.$$

But since $d\alpha_\theta/dp \neq 0$, we see that $S'_\theta(\alpha_\theta) = 0$ and thus $\partial\Phi/\partial\alpha_\theta = S'_\theta(\alpha_\theta) = 0$. Hence (3.4) becomes

$$\partial L_\theta/\partial\theta = \frac{\partial\Phi}{\partial\theta} = -\frac{1}{2}\int_0^{\alpha_\theta(p_\theta)} (v^2 - \alpha_\theta^2)(F_\theta(\alpha_\theta(p_\theta)) - F_\theta(v))^{-3/2}\, dv > 0.$$

This completes the proof.

Let $\bar{\theta}$ be defined as above so that if

$$0 \leqslant \theta < \bar{\theta}, \tag{3.5}$$

the function f defined by (3.2), has three real roots $0 < a_\theta < b_\theta$, and satisfies (3.3). Let $L > L_{\bar{\theta}}$; then $L > L_\theta$ for all θ satisfying (3.5). Thus for such θ, (3.1) admits precisely two non-constant solutions. *From now on, in Sections 3 and 4, we take $L > L_\theta$, and fix this value of L.*

For θ satisfying (3.5), and $L > L_\theta$, let v_1, v_2 denote the two non-constant solutions of (3.1); (we now drop the dependence on θ in these solutions). The solution v_1 is unstable, and its' generalized Morse index is $h(v_1) = \Sigma^1$, the pointed one-sphere; in fact v_1 has a 1-dimensional unstable manifold. Similarly, $h(v_2) = \Sigma^0$, and v_2 is an attractor. To each of these solutions, there are associated the corresponding (vector) solutions of (2.3), (2.4_D); namely,

$$V_1(x) = (1, \delta/\gamma)\, v_1(x), \quad V_2(x) = (1, \delta/\gamma)\, v_2(x), \quad |x| \leqslant L. \tag{3.6}$$

We denote the solution (0, 0) by V_0; then if we linearize (1.1), (2.1_D) about V_j ($j = 0, 1,$ or 2), we obtain the following (eigenvalue) equations on the interval $|x| < L$:

$$\begin{aligned} \lambda w &= w'' + f'(v_j)\, w - z, \qquad w(\pm L) = 0 \\ \lambda z &= \delta w - \gamma z. \end{aligned} \tag{3.7_j}$$

In order to study the stability of our solutions, we must consider the spectra of the operators

$$A_j \colon \begin{pmatrix} w \\ z \end{pmatrix} \to \begin{pmatrix} w'' + f'(v_j)\, w - z \\ \delta w - \gamma z \end{pmatrix}, \qquad j = 0, 1, 2, \tag{3.8_j}$$

defined on the space $C_0^{2+\alpha}(|x| < L) \times C^\alpha(|x| < L)$ into $C^\alpha(|x| < L) \times C^\alpha(|x| < L)$, where the subscript 0 denotes the fact that $w(\pm L) = 0$. Let $B_j \colon C_0^{2+\alpha}(|x| < L) \to C^\alpha(|x| < L)$ be the operator defined by

$$B_j \colon w \to w'' + f'(v_i)\, w - \delta w/\gamma, \qquad j = 0, 1, 2. \tag{3.9}_j$$

Note that $-\gamma$ is not an eigenvalue of A_j; this is clear from (3.7_j). Thus using (3.7_j), if λ is an eigenvalue of A_j, we have

$$\begin{aligned} \lambda w &= w'' + f'(v_j)\, w - \delta w/(\lambda + \gamma) \\ &= w'' + f'(v_j)\, w - \delta w/\gamma + [\delta w/\gamma - \delta w/(\lambda + \lambda)] \\ &= w'' + f'_\theta(v_j)\, w + \lambda \delta w/\gamma(\lambda + \gamma), \end{aligned}$$

and $w(\pm L) = 0$. Thus we obtain

$$\lambda k w = w'' + f'_\theta(v_j)\, w, \qquad k = 1 - \delta/\gamma(\lambda + \gamma). \tag{3.10}_j$$

From this it follows that if λ is an eigenvalue of A_j, then λk must be real. Furthermore, we have the following lemma.

LEMMA 3.2. *0 is not an eigenvalue of* A_j, $i = 0, 1, 2$.

The proof of this follows from (3.10_j) and the corresponding statement for the scalar operators $d^2/dx^2 + f'_\theta(v_j)$, with homogeneous Dirichlet boundary conditions; see [11].

We shall base our analysis of the eigenvalues of the operators A_j on the equation (3.10_j). Thus, if $\lambda = \alpha + i\beta$ is an eigenvalue of A_j, then (3.10_j) holds. An easy computation gives both of the following equations

$$\operatorname{Re}(\lambda k) = \alpha\left[1 - \frac{\delta}{\gamma}\frac{(\alpha+\gamma)}{\beta^2+(\alpha+\gamma)^2}\right] - \frac{\delta\beta^2}{\gamma}\frac{1}{\beta^2+(\alpha+\gamma)^2} \tag{3.11}$$

$$\operatorname{Im}(\lambda k) = \beta[(\alpha+\gamma)^2 + \beta^2 - \delta]/[(\alpha+\gamma)^2 + \beta^2]. \tag{3.12}$$

Since λk is real, one of the following must hold; namely,

$$\beta = 0 \qquad \text{or} \qquad \beta^2 + (\alpha+\gamma)^2 = \delta. \tag{3.13}$$

LEMMA 3.3. (i) *A_0 has no eigenvalues in* $\operatorname{Re} z \geqslant 0$.

(ii) *If the parameters δ and γ satisfy the inequality*

$$\gamma^2 \geqslant \delta \tag{3.14}$$

then A_1 and A_2 have no eigenvalues on $\operatorname{Re} z = 0$, *and A_2 has no eigenvalues in* $\operatorname{Re} z > 0$.

Proof. We consider first A_0. Using (3.10_0), we have

$$w'' + f'(0)\, w - \delta w/\gamma = \lambda k w, \qquad w(\pm L) = 0,$$

where k is given in (3.10_0). It follows that the quantity

$$\lambda k - f'(0) + \delta/\gamma \equiv \tilde{\lambda}$$

is negative, since it is an eigenvalue of the operator $D^2\colon C_0^{2+\alpha}(|x| < L) \to C^\alpha(|x| < L)$. Thus

$$\lambda k = \mu < 0.$$

Now if we use the expression for k given in (3.10_j), we get $\lambda[1 - \delta/\gamma(\lambda+\gamma)] = \mu$, and solving for λ gives

$$\begin{aligned} 2\lambda &= (\mu - \gamma + \delta/\gamma) \pm \sqrt{(\mu - \gamma + \delta/\gamma)^2 + 4\mu\gamma} \\ &= (\tilde{\lambda} + f'(0) - \gamma) \pm \sqrt{(\tilde{\lambda} + f'(0) - \gamma)^2 + 4\mu\gamma}. \end{aligned}$$

Thus $\operatorname{Re} \lambda < 0$, and this proves (i).

Now suppose that (3.14) holds. If $\lambda = \alpha + i\beta$ is an eigenvalue of A_1 or A_2, and $\alpha = 0$, then from (3.13) we see that $\beta = 0$, and this contradicts Lemma 3.2. Finally, let λ be an eigenvalue of A_2, where $\alpha > 0$. Then since

$$(\alpha + \gamma)^2 + \beta^2 > \gamma^2 + \beta^2 \geqslant \gamma^2 \geqslant \delta$$

we see from (3.13) that $\beta = 0$. Hence λ is real, $\lambda = \alpha > 0$. Thus

$$k = (\alpha\gamma + \gamma^2 - \delta)/\gamma(\alpha + \gamma) > 0,$$

so that $\lambda k > 0$. It follows from (3.10_2) and the fact that v_2 is a stable non-degenerate solution of the scalar equation (2.9), that $w \equiv 0$. Thus, from (3.7_2), we find $z = 0$ and λ is not an eigenvalue of A_2. ∎

We shall now show that the operators A_i, $i = 0, 1, 2$, have no continuous spectrum. Since we are considering these operators on a finite domain, $|x| < L$, if the second equation in (1.1) had a non-zero diffusion term εu_{xx}, this conclusion would follow from standard theorems. To obtain the result in our case, we take advantage of the special structure of our equations.

THEOREM 3.4. *The operators A_i, $i = 0, 1, 2$, have pure point spectrum.*

Proof. Let $\lambda = \alpha + i\beta$. We shall show the following two things; namely, that except for a discrete set of λ,

(i) $(A_i - \lambda I)^{-1}$ is everywhere defined, and

(ii) $(A_i - \lambda I)^{-1}$ is a closed operator.

From (i) it follows that the domain of $(A_i - \lambda I)^{-1}$ is closed, and since (ii) implies that $(A_i - \lambda I)^{-1}$ has closed graph, the closed graph theorem shows that $(A_i - \lambda I)^{-1}$ is bounded. Thus λ is not in the continuous spectrum of A_i. (Actually (ii) follows from (i). To see this, suppose that $V_n \to V$ in C^α and $(A_i - \lambda I)^{-1} V_n \to W$ in $C_0^{2+\alpha}$. Since $(A_i - \lambda I)$ is continuous and everywhere defined on $C_0^{2+\alpha}$, $V_n \to (A_i - \lambda I) W$ in C^α. Therefore $(A_i - \lambda I) W = V$ so from (i) $W = (A_i - \lambda I)^{-1} V$, and this gives (ii).)

To prove the theorem, it suffices to show (i). Thus suppose $(\phi, \psi) \in C^\alpha \times C^\alpha$; we must show that the equations

$$\begin{aligned} w'' + f'(v_i)\, w - z - \lambda w &= \phi, \qquad w(\pm L) = 0 \\ \delta w - \lambda z - \lambda z &= \psi \end{aligned} \tag{3.15}$$

are solvable for $(w, z) \in C_0^{2+\alpha} \times C^\alpha$. Note that we may assume $\lambda + \gamma \neq 0$.

Using the second equation in (3.15), we find

$$z = (\delta w - \psi)/(\gamma + \lambda), \tag{3.16}$$

and substituting this in the first equation gives $w''+f'(v_i)\,w-\delta w/(\gamma+\lambda)-\lambda w=\phi-\psi/(\gamma+\lambda)$. This last equation can be rewritten as

$$w''+f'_\theta(v_i)\,w+\eta w=\phi-\psi/(\gamma+\delta), \qquad w(\pm L)=0, \tag{3.17}$$

where

$$\eta=\delta/\gamma-\delta/(\gamma+\lambda)-\lambda. \tag{3.18}$$

Now if $-\eta$ is not in the spectrum of the operator B_i (see (3.9)), then (3.17) can be solved for w, and (3.16) gives the corresponding z. Thus (3.17) is solvable for all η which do not lie in the discrete set $\mathrm{sp}(-B_i)$. If we solve (3.18) for λ, we find

$$2\lambda=(\delta/\gamma-\gamma-\eta)\pm\sqrt{(\delta/\gamma-\gamma-\eta)^2-4\gamma\eta}\equiv 2\lambda(\eta). \tag{3.19}$$

It follows that if λ is not in the set $\lambda(\mathrm{sp}(B_i))$, then $\eta\notin\mathrm{sp}(-B_i)$ and (3.17) is solvable. This completes the proof since $\mathrm{sp}(-B_i)$ is a countable, discrete set. ∎

As a consequence of the last two results, we have the following theorem; see [3, 10].

THEOREM 3.5. *V_0 is an isolated invariant set for the system* (1.1), (2.1_{D}), *and $h(V_0)=\Sigma^0$. If* (3.14) *holds, the same statement is valid for V_2. In particular, both of these steady state solutions are stable, i.e., they are attractors for the equation* (1.1), (2.1_{D}).

In order to compute the index of V_1, indeed to even show that it is an isolated invariant set, we shall show that if (3.14) holds, Eqs. (1.1), (2.1_{D}) are gradient-like. This fact will also enable us to describe the entire solution space, for these parameter values.

Let $\tilde\delta=\delta/r$, and rewrite (1.1) as

$$\begin{aligned} v_t&=(v_{xx}+f(v)-\tilde\delta v)+(\tilde\delta v-u), \qquad v(\pm L,t)=0,\\ u_t&=\gamma(\tilde\delta v-u). \end{aligned}$$

The function we consider is

$$\begin{aligned} \mathscr{L}(u,v)&=\int_{-L}^{L}(-v_x^2/2+F(v)-\tilde\delta v^2+uv-u^2/2\tilde\delta)\,dx\\ &=\int_{-L}^{L}[(-v_x^2/2+F(v)-\tilde\delta v^2/2)+(uv-\tilde\delta v^2/2-u^2/2\tilde\delta)]\,dx, \end{aligned} \tag{3.20}$$

where $F' = f$. Now we compute

$$\begin{aligned}\partial\mathscr{L}/\partial t = \int_{-L}^{L} \{ &(v_{xx} + f(v) - \bar{\delta} v)^2 + (v_{xx} + f(v) - \bar{\delta} v)(\bar{\delta} v - u) \\ &+ u(v_{xx} + f(v) - \bar{\delta} v) + u(\bar{\delta} v - u) + \gamma v(\bar{\delta} v - u) \\ &- \bar{\delta} v(v_{xx} + f(v) - \bar{\delta} v) - \bar{\delta} v(\bar{\delta} v - u) - \gamma u(\bar{\delta} v - u)/\bar{\delta} \}\, dx \\ = \int_{-L}^{L} \{ &(v_{xx} + f(v) - \bar{\delta} v)^2 + (\bar{\delta} v - u)(u + v\gamma - \bar{\delta} v - \gamma u/\bar{\delta}) \}\, dx \\ = \int_{-L}^{L} \{ &(v_{xx} + f(v) - \bar{\delta} v)^2 + (\gamma^2 - \delta)(\bar{\delta} v - u)^2/\delta \}\, dx.\end{aligned}$$

We thus have proved the following lemma.

LEMMA 3.6. *If* (3.14) *holds, then the system* (1.1), (2.1_{D}) *is gradient-like*[1] *with respect to the function* $\mathscr{L}$ *defined in* (3.20).

As we have noted in Proposition 2.1, all solutions of (1.1), (2.1_{D}) tend to the rectangle R in $(v-u)$-space; i.e., R is a global attractor for solutions of (1.1), (2.1_{D}). Under these circumstances, we have the following theorem.

THEOREM 3.7. *Assume that* (3.14) *holds:*

A. *There are solutions* $U_i(x, t) = (v_i(x, t), u_i(x, t))$, $i = 1, 2$, *of* (1.1), (2.1_{D}) *satisfying*

$$\lim_{t \to -\infty} U_1(x, t) = V_1(x), \qquad \lim_{t \to \infty} U_1(x, t) = V_0(x)$$

$$\lim_{t \to -\infty} U_2(x, t) = V_1(x), \qquad \lim_{t \to \infty} U_2(x, t) = V_2(x)$$

uniformly in $|x| \leqslant L$.

B. $h(V_1) = \Sigma^1$, *and* V_1 *has a* 1-*dimensional unstable manifold.*

Proof. The proof of A. is similar to a theorem we have given in [3]; see also [10]. Moreover, as in these references, we can show that $h(V_1)$ has the cohomology of a one-sphere. To complete the proof of B, it suffices to show that the spectrum of the operator A_1 has a finite number of eigenvalues with positive real part in Re $z \geqslant 0$. Then since V_1 is non-degenerate (see Lemma 3.2), it follows that $h(V_1) = \sum^p$ for some p; whence $p = 1$, and V_1 has a 1-dimensional unstable manifold.

[1] Thus $\partial\mathscr{L}/\partial t \geqslant 0$ and $\partial\mathscr{L}/\partial t = 0$ precisely on the rest points.

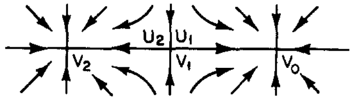

FIGURE 3

Let $\lambda = \alpha + i\beta$, $\alpha \geqslant 0$, be an eigenvalue of A_1. As we have seen above (Eqs. (3.10 ff), the quantity λk is real, where k is defined in (3.10). Thus $\mathrm{Im}(\lambda k) = 0$ so (3.12) and (3.14) imply that $\beta = 0$. We can write (3.10) as

$$B_1 w = w'' + f'_\theta(v_1)\, w = \tilde{\lambda} w, \qquad w(\pm L) = 0,$$

where $\tilde{\lambda} = \alpha[1 - \delta/\gamma(\alpha + \gamma)]$. Since B_1 has precisely one positive eigenvalue, $\partial\tilde{\lambda}/\partial\alpha \geqslant 0$, $\partial^2\tilde{\lambda}/\partial\alpha^2 > 0$, and $\tilde{\lambda} = 0$ when $\alpha = 0$, we see that A_1 has precisely one eigenvalue in $\mathrm{Re}\, z > 0$; it is in fact real. This completes the proof. ∎

The results obtained in this section enable us to completely describe the entire solution set of (1.1), (2.1_D) in the case where (3.14) holds. In fact, we know that all solutions must tend to one of the rest points V_i, $i = 0, 1, 2$, as $t \to +\infty$, and that there are two solutions, $U_1(x, t)$, $U_2(x, t)$, which "connect" the rest points V_1, V_0 and V_1, V_2, respectively (cf. Theorem 3.7). There are no other solutions. Thus, the complete solution set can be "depicted" as in Fig. 3.

We can depict the relevant region in parameter space; see Fig. 4. Note that the region in question; i.e., the region where (3.14) is valid, is the shaded region in Fig. 3. It is interesting to observe that in the region marked T in Fig. 4, V_0 is the only steady state solution of (1.1), (2.1_D) (because $\theta > \bar{\theta}$). Since the region T lies in $\gamma^2 \geqslant \delta$, the equations are still gradient-like. It follows that all solutions must tend to V_0; this holds for all L. Thus V_0 is a global attractor. This fact is noteworthy since it cannot be obtained by "invariant rectangle" techniques. It relies on a deeper fact; namely the existence of the gradient like function $\mathscr{L}$ in the parameter region T.

We close this section by demonstrating an alternate way of computing the indices of V_0 and V_2, in the case where (3.14) holds. This method is

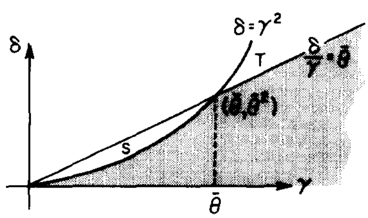

FIGURE 4

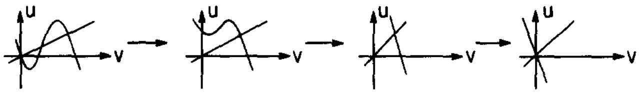

FIGURE 5

very much the spirit of the method which we described in [3], for the scalar equation. It relies again on the existence of the gradient-like function L. Thus, since the V_i are isolated rest points, they are isolated invariant sets, [2], and so they have well-defined indices, which are stable under continuation. Consider, for example, V_2. We successively continue Eqs. (1.1), by deforming f, as depicted in Fig. 5. That is, we continue Eqs. (1.1), (2.1_D) to the linear equations as depicted in Fig. 5(iv); these have the form

$$\begin{aligned} v_t &= v_{xx} - \alpha v - \beta u, \qquad v(\pm L, t) = 0 \\ u_t &= \delta v - \gamma u, \end{aligned} \tag{3.21}$$

where the constants α and β are positive and $\alpha/\beta > \delta/\gamma$. Under this deformation V_2 continues to the zero solution of (3.21), and $h(V_2) = h(0)$. Since 0 is a global attractor for (3.21) (see [9], for example), we have $h(V_2) = \Sigma^0$. Similarly, we can show that $h(V_0) = \Sigma^0$.

4. Bifurcation of Solutions of the Dirichlet Problem

In this section we shall allow the parameters γ and δ to violate (3.14). We will see that the attractor V_2 becomes unstable, and that the spectrum of A_2 picks up exactly two eigenvalues in $\operatorname{Re} z > 0$. In this case $h(V_2) = \Sigma^2$, a "Hopf bifurcation" occurs, and a periodic solution appears near V_2. That is, we shall obtain a solution $U = (v, u)$ of (1.1), (2.1_D) which satisfies

$$U(x, t+T) = U(x, t), \qquad |x| \leqslant L,\ t \geqslant 0,$$

for some $T = T(\gamma, \delta)$.

We begin by studying the eigenvalues of the operator A_2, when (3.14) fails. As in Section 3, if λ is an eigenvalue for A_2, then $(3.10)_2$ holds, and

$$\lambda k = \bar{\mu}, \tag{4.1}$$

where $\bar{\mu}$ is an element in the spectrum of B_2. If $\mu = \mu(\delta/\gamma)$ is the largest eigenvalue of B_2, then $\bar{\mu} \leqslant \mu < 0$, since v_2 is a stable solution of (2.7), (2.1_D). Using the expression for k given in (3.10_2), we can solve (4.1) for λ to get

$$2\lambda = (\bar{\mu} + \delta/\gamma - \gamma) \pm \sqrt{(\bar{\mu} + \delta/\gamma - \gamma)^2 + 4\bar{\mu}\gamma}. \tag{4.2}$$

Note that as before, if $\gamma^2 \geqslant \delta$ (4.2) shows that λ has negative real parts. Thus, as (3.14) is violated, we see that λ crosses the imaginary axis when the function

$$G(\gamma, \delta) \equiv \mu + \delta/\gamma - \gamma = 0.$$

Consider next the equation $G(\gamma, \delta) = 0$. Since $G_\delta = (\mu' + 1)/\gamma$, and $G_\gamma = -\delta(\mu' + 1)/\gamma^2 - 1$, we see that $\nabla G \neq 0$, so that the equation $G(\gamma, \delta) = 0$ defines a one-manifold. Note that if $0 < \gamma < \bar{\theta}$ then since $\mu(\bar{\theta}) = 0$, $G(\gamma, \gamma\bar{\theta}) = \mu(\bar{\theta}) + \bar{\theta} - \gamma = \bar{\theta} - \gamma > 0$. Thus $G > 0$ on the line $\delta = \gamma\theta$.

Also, if $G = 0$, then $\delta > \gamma^2$ so that the curve $G = 0$ lies above the parabola $\delta = \gamma^2$, and since $G(\bar{\theta}, \bar{\theta}^2) = \mu(\bar{\theta}) = 0$, we see that the branch of $G = 0$ of interest to us lies in the "moon-shaped" region S depicted in Fig. 4.

Now let γ be such that $0 < \gamma < \bar{\theta}$; since $G(\gamma, \gamma^2) < 0$, we can find a point $P = (\tilde{\gamma}, \tilde{\delta})$, $0 < \tilde{\gamma} < \bar{\theta}$ for which the curve $G = 0$ meets the line $\gamma = \tilde{\gamma}$ transversally at P and $G(\tilde{\gamma}, \delta + \varepsilon) > 0$ while $G(\tilde{\gamma}, \tilde{\delta} - \varepsilon) < 0$ for some $\varepsilon > 0$. This gives the following theorem.

THEOREM 4.1. *There is a point* $P = (\tilde{\gamma}, \tilde{\delta})$, $0 < \tilde{\gamma} < \bar{\theta}$, $0 < \tilde{\delta} < \bar{\theta}\tilde{\gamma}$ *(i.e., a point in region S in Fig. 4) such that* $G(\tilde{\gamma}, \tilde{\delta}) = 0$, *and* $G_\delta(\tilde{\gamma}, \tilde{\delta}) > 0$. *At P, a time periodic solution of* (1.1), (2.1_{D}) *bifurcates out of* V_2, *as* δ *increases across* $\tilde{\delta}$, *on the line* $\gamma = \tilde{\gamma}$. *Near P, for* $\delta > \tilde{\delta}$, *the spectrum of the operator* A_2 *has exactly two eigenvalues with positive real parts, and* $h(V_2) = \Sigma^2$.

In other words, as δ increases across $\tilde{\delta}$ along the line $\gamma = \tilde{\gamma}$, a "Hopf bifurcation" occurs at P.

Proof. Using (4.2), we see that as δ increases through $\tilde{\delta}$ along $\gamma = \tilde{\gamma}$, the spectrum of A_2 picks up exactly two eigenvalues with positive real parts; thus $h(V_2) = \Sigma^2$ for these parameter values. Since $G_\delta(\tilde{\gamma}, \tilde{\delta}) > 0$, a Hopf bifurcation occurs (see e.g. [6, pp. 250–257; or 7, pp. 233 ff] and a periodic solution bifurcates out of V_2. ∎

In a future publication, we shall allow the parameter δ to further increase along the line $\gamma = \tilde{\gamma}$, and we shall study the corresponding solution set. It is thus of interest to close this section with the following result.

PROPOSITION 4.2. *The operator* A_1 *always has at least one positive real eigenvalue.*

Proof. In view of Theorem 3.7, it suffices to assume that $\delta > \gamma^2$. If μ is the positive eigenvalue of B_1, and λ is defined by

$$2\gamma = (\mu + \delta/\gamma - \gamma) + \sqrt{(\mu + \delta/\gamma - \gamma)^2 + 4\mu\gamma},$$

then $\lambda > 0$ and

$$\lambda\left(1 - \frac{\delta}{\gamma}\frac{1}{(\lambda+\gamma)}\right) = \mu.$$

Thus, if w is the eigenfunction of B_1 corresponding to μ, and $z = \delta w/(\lambda+\gamma)$, then we see that (z, w) is an eigenfunction of A_1 corresponding to the eigenvalue λ. ∎

5. The Neumann Problem—Bifurcation of Solutions

We briefly consider Eqs. (1.1) with homogeneous Neumann boundary conditions (2.1_{N}). In particular, we are interested in the existence and bifurcation of the steady state solutions; that is, solutions to the problem (2.3), (2.4_{N}). As we have seen earlier, the existence problem reduces to that of Eqs. (2.3′), (2.4_{N}). In order that this problem has solutions, we choose δ/γ so small that the polynomial (3.2) has three real roots, $0 < a < b$.

As was shown in [12], the global bifurcation diagram for the non-constant stationary solutions can be depicted as in Fig. 6. The stationary solutions are obtain by studying the "time map" associated with the stationary solutions of the Neumann problem (see [12]). These solutions are all non-degenerate in the sense that 0 is not in the spectrum of their linearizations; hence they do not undergo "secondary bifurcation." The solutions depicted in Fig. 6 all bifurcate out of the constant solution $u \equiv a$. Furthermore, the two other constant solutions $u \equiv 0$ and $u \equiv b$, are stable non-degenerate solutions, and do not undergo bifurcation. Referring to Fig. 6, the points L_n correspond to those L values in which the linearized equations about $u \equiv a$ pick up a positive eigenvalue. Moreover, if $L_n < L < L_{n+1}$ ($L_0 = 0$), $h(a) = \Sigma^{n+1}$, and along the branch out of L_n, each of the two (symmetric) solutions v_n', and v_n'' have index Σ^n; these correspond to solutions in $(x-u)$ space which have $(n-1)$ internal extrema.

From these remarks, we see that we can also obtain all of the stationary solutions for the system (1.1), (2.1_{N}); they are of the form

$$(1, \delta/\gamma)\, v,$$

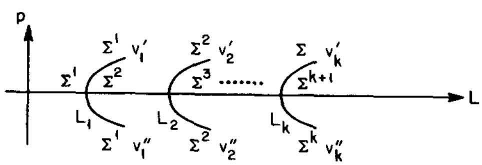

Figure 6

where v solves (2.3′), (2.4$_{\mathrm{N}}$). In the region $\gamma^2 \geqslant \delta$, it is easily checked that the function L, defined in (3.2) is a global Lyapunov function for our system. This yields the following theorem.

THEOREM 5.1. *If* (3.14) *holds, and* $L_n < L < L_{n+1}$, *then*

$$h[(1, \delta/\gamma)\, a] = \Sigma^{n+1}, \quad h[(1, \delta/\gamma)\, v_n'] = h[(1, \delta/\gamma)\, v_n''] = \Sigma^n.$$

Moreover, there are solutions $U_i^n(x, t)$, $i = 1, 2$ *of* (1.1), (2.1$_{\mathrm{N}}$), *which "connect" these critical points; i.e.,*

$$\lim_{t \to -\infty} U_i^n(x, t) = (1, \delta/\gamma)\, a, \qquad i = 1, 2,$$

$$\lim_{t \to \infty} U_1^n(x, t) = (1, \delta/\gamma)\, v_n',$$

and

$$\lim_{t \to \infty} U_2^n(x, t) = (1, \delta/\gamma)\, v_n'',$$

uniformly on $|x| \leqslant L$.

Proof. The proof is virtually identical to the scalar case (see [10]), provided that we can show $h[(1, \delta/\gamma)\, a] = \Sigma^{n+1}$. As in the proof of Theorem 3.4, it is easy to show that the linearized operator around $(1, \delta/\gamma)\, a$ has pure point spectrum; thus we shall show that this operator has exactly $(n+1)$ eigenvalues in $\operatorname{Re} z \geqslant 0$.

To this end, consider the eigenvalue problem

$$w'' + f'(a)\, w - z = \lambda w, \qquad w'(\pm L) = 0$$
$$\delta w - \gamma z = \lambda z,$$

where $\lambda = \alpha + i\beta$. Since $\lambda = -\gamma$ cannot be an eigenvalue, we obtain the equation

$$w'' + (f'(a) - \delta/\gamma)\, w = \lambda k w,$$

where k is defined in (3.10$_j$). Since $\operatorname{Im}(\lambda k) = 0$ and (3.14) holds, $\beta = 0$ so αk is an eigenvalue of the operator

$$B = d^2/dx^2 + (f'(a) - \delta/\gamma),$$

with homogeneous Neumann boundary conditions on $x = \pm L$. From the results of [4], B has exactly $(n+1)$ positive eigenvalues $\mu_1, \ldots, \mu_{n+1}$, and 0 is not an eigenvalue of B. If now μ is any eigenvalue of B, then $\alpha k = \mu$ implies $\alpha = \alpha_+$ or α_-, where

$$2\alpha_{\pm} = (\mu + \delta/\gamma - \gamma) \pm \sqrt{(\mu + \delta/\gamma - \gamma)^2 + 4\mu\gamma}.$$

If $\mu < 0$, then $\alpha_{\pm} < 0$, while if $\mu = \mu_i$, then $\alpha_+ > 0$, and $\alpha_- < 0$. ∎

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